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Eigenstates of paraparticle creation operators

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Abstract. Eigenstates of the parabose and parafermi creation operators are constructed. In the Dirac contour representation, the parabose eigenstates correspond to the dual vectors of the parabose coherent states. In order $p = 2$, conserved-charge parabose creation operator eigenstates are also constructed. The contour forms of the associated resolutions of unity are obtained.

1. Introduction

In quantum theories, the usual Hilbert space can accomodate eigenstates of bosonic annihilation operators, which are referred to as 'coherent states', whose expansion coefficients in the number basis of the Hilbert space are ordinary functions. If one enlarges this space by allowing not only ordinary functions, but also distributions as the expansion coefficients of states in the number basis, then the enlarged Hilbert space will accomodate eigenstates of bosonic creation operators. Eigenstates of the ordinary bose creation operator were constructed in [1] using Heitler's contour integral form of the δ-function [2]. These eigenstates correspond to the dual vectors of the coherent states in Dirac's contour representation of boson systems [3, 1, 4]. It is natural to investigate whether such creation operator eigenstates can also be constructed for paraparticles [5, 6]. Parabose coherent states were proposed in [7], parafermi coherent states in [6], and recently, parabose squeezed states in [8]. In local relativistic quantum field theory, particles obeying parastatistics with $p > 1$ are allowed and they might be produced at the higher energies of the new/future colliders.

In section 2 of this paper, the eigenstates for the parabose creation operator are constructed. Heitler's form of the δ-function is used, and the expansion coefficients for these eigenstates in the parabose number basis are distributions. It is worthwhile recalling that there is another kind of enlargement of the usual Hilbert space that occurs when the expansion coefficients are paragrassman numbers. In this case, eigenstates of parafermi creation operators can be constructed. In section 3, paragrassman numbers are used in the construction of the eigenstates of the parafermi creation operator. In the number basis, the expansion coefficents for the f eigenstates and for the $f^\dagger$ eigenstates are paragrassman numbers. In this enlarged Hilbert space, we also discuss various inner products of the eigenstates of parafermi annihilation and creation operators, and the completeness relations of these eigenstates. Lastly, in section 4, the conserved-charge parabose creation operator eigenstates are constructed for the two-mode

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parabose system in order $p = 2$. In each section, the respective contour forms of the resolution of unity are derived.

There are two open questions implicit in the present analysis. (i) While the physical and practical significance of the ordinary coherent states (the eigenstates of the boson annihilation operators) is well known, to date such understanding is lacking for eigenstates of the parabose $a^\dagger$ and parafermi $f^\dagger$ creation operators. (ii) What (rigorously) is the enlarged Hilbert space which contains these creation operator eigenstates? In particular, the parabose eigenstate $|z\rangle'$ of $a^\dagger$ in (9) below is given as a formal expansion without discussion of the convergence of the series and normalization of the states. In regard to a formulation of a full distribution theoretic framework for these improper eigenstates, it is noteworthy that the creation operator eigenstates, e.g. $|z\rangle'$ of $a^\dagger$, do correspond to the parabose coherent state's eigenbra $\langle\alpha|$ of $a^\dagger$ in the Dirac contour representation (see equations (11), (13), (40), (52) below).

2. Eigenstates of the parabose $a^\dagger$ operator

For a single-mode parabose system, the number basis is

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle, \qquad N_B|n\rangle = n|n\rangle \tag{1}$$

where N_B is the number operator $N_B = \frac{1}{2}\{a^\dagger, a\} - \frac{p}{2}$ with p the order of the parastatistics. The eigenvalue of the deformed parabose number operator $[N_B]$ is

$$[n] = n + \frac{p-1}{2}(1 - (-)^n) \tag{2}$$

with $[n]! = [n][n-1]\cdots[1]$, $[0]! \equiv 1$. The parabose number states satisfy

$$a|n\rangle = \sqrt{[n]}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle. \tag{3}$$

In this basis, the unnormalized coherent states [7] are

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle = E(za^\dagger)|0\rangle \qquad E(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{4}$$

with $a|z\rangle = z|z\rangle$.

We denote the eigenstate of the creation operator $a^\dagger$ by a 'primed' ket $|z\rangle'$,

$$a^\dagger|z\rangle' = z^*|z\rangle' \tag{5}$$

and expand it $|z\rangle' = \sum_{n=0}^{\infty} c_n(z^*)|n\rangle$ in the number basis. By (3), the resulting recursion relations are

$$c_0\, z^* = 0 \qquad c_1\, z^* = \sqrt{[1]}\, c_0 \qquad c_n\, z^* = \sqrt{[n]}\, c_{n-1} \tag{6}$$

or $c_n = \left(\sqrt{[n]!}/(z^*)^n\right)c_0$. By the Cauchy integral formula for an analytic function $f(z^*)$, or alternatively by use of Heitler's δ-function [2] in the contour integral†, it follows [1] that

$$c_0 = \left.\frac{1}{z^*}\right|_{C^*} = \delta(z^*) \tag{7}$$

$$c_n = \left.\frac{\sqrt{[n]!}}{(z^*)^{n+1}}\right|_{C^*} = \frac{(-)^n\sqrt{[n]!}}{n!}\, \delta^{(n)}(z^*). \tag{8}$$

† Note that

$$f(0) = \oint_{C^*} \frac{\mathrm{d}z^*}{2\pi \mathrm{i}}\, f(z^*)\delta(z^*).$$

The nth derivative is

$$f^{(n)}(0) = n! \oint_{C^*} \frac{\mathrm{d}z^*}{2\pi \mathrm{i}}\, \frac{f(z^*)}{(z^*)^{n+1}} = (-)^n \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\, \delta^{(n)}(x).$$

The notation $|_{C^*}$ means that the subsequent integration over z^* must be over the anticlockwise contour C^* enclosing the origin in the complex z^* plane.

So, the eigenstate of $a^\dagger$ is

$$|z\rangle' = \sum_{n=0}^{\infty} \frac{(-)^n\sqrt{[n]!}}{n!}\,\delta^{(n)}(z^*)|n\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{[n]!}}{(z^*)^{n+1}}|n\rangle\bigg|_{C^*} = \sum_{n=0}^{\infty} \frac{(a^\dagger)^n}{(z^*)^{n+1}}|0\rangle\bigg|_{C^*} \tag{9}$$

and, formally (in the number basis)

$$|z\rangle' = \frac{1}{z^* - a^\dagger}|0\rangle\bigg|_{C^*}. \tag{10}$$

Note that the action of integer powers of $(a^\dagger)^m$ removes the contribution of the number states $n < m$ in this expansion; for example

$$a^\dagger|z\rangle' = a^\dagger\,\frac{1}{z^* - a^\dagger}|0\rangle\bigg|_{C^*} = z^*\,\frac{1}{z^* - a^\dagger}|0\rangle\bigg|_{C^*} - |0\rangle\big|_{C^*} = z^*\,|z\rangle'$$

since the $|0\rangle\big|_{C^*}$ term gives no contribution because of the subsequent contour integration.

In the Dirac contour representation, the dual vector $\langle\alpha|$ of the parabose coherent state $|\alpha\rangle$, given above in (4), [10] is

$$\langle\alpha| = \sum_{n=0}^{\infty}\langle n|\frac{(\alpha^*)^n}{\sqrt{[n]!}} \quad\to\quad \sum_{n=0}^{\infty}\frac{\sqrt{[n]!}}{z^{n+1}}\frac{(\alpha^*)^n}{\sqrt{[n]!}} = \frac{1}{z-\alpha^*} \qquad (|z| > |\alpha|). \tag{11}$$

with $\langle\alpha|a^\dagger = \langle\alpha|\alpha^*$. Thus, the eigenstate $|z\rangle'$ of $a^\dagger$ in the number basis corresponds to the parabose coherent state's eigenbra $\langle\alpha|$ of $a^\dagger$ in the Dirac contour representation.

The inner product of the unnormalized parabose coherent state $|w\rangle$ and the eigenstate $|z\rangle'$ is given by

$$\begin{aligned}\langle w|z\rangle' &= \sum_{n,m=0}^{\infty}\langle n|\frac{(w^*)^n}{\sqrt{[n]!}}\frac{\sqrt{[m]!}}{(z^*)^{m+1}}|m\rangle\bigg|_{C^*} = \sum_{n=0}^{\infty}\frac{(w^*)^n}{(z^*)^{m+1}}\bigg|_{C^*}\\ &= \frac{1}{z^*-w^*}\bigg|_{C^*} = \delta(z^*-w^*) \qquad (|z| > |w|)\end{aligned} \tag{12}$$

and so they satisfy the 'contour form' of the resolution of unity (see [3])

$$\oint_{C^*}\frac{\mathrm{d}z^*}{2\pi\mathrm{i}}\,|z\rangle'\langle z| = \sum_{n,m=0}^{\infty}\oint_{C^*}\frac{\mathrm{d}z^*}{2\pi\mathrm{i}}\,\frac{\sqrt{[n]!}}{(z^*)^{n+1}}|n\rangle\langle m|\frac{(z^*)^m}{\sqrt{[m]!}} = \sum_{n=0}^{\infty}|n\rangle\langle n| = I. \tag{13}$$

Remark. This resolution of unity can be used to derive a contour integral expressions for the parabose Hermite polynomials [11]: From (13), the parabose coordinate eigenstate $|x\rangle$ can be written as

$$|x\rangle = \sum_{n=0}^{\infty}|n\rangle\frac{\sqrt{[n]!}}{2\pi\mathrm{i}}\oint_{C^*}\frac{\mathrm{d}z^*}{(z^*)^{n+1}}\langle z|x\rangle \tag{14}$$

where $\langle x|z\rangle$ is the wavefunction of the parabose coherent state in the parabose coordinate representation. We consider

$$\langle x|z\rangle = \frac{1}{x}\langle x|\hat{x}|z\rangle = \frac{1}{x\sqrt{2}}\langle x|(a+a^\dagger)|z\rangle. \tag{15}$$

From (3), cf equation (14) for the parabose deformed derivative D/Dz in [10],

$$a^\dagger|z\rangle = \frac{\partial}{\partial z}|z\rangle + \left(\frac{p-1}{2z}\right)|z\rangle - \left(\frac{p-1}{2z}\right)|-z\rangle \tag{16}$$

so (15) gives

$$\frac{\partial}{\partial z}\langle x|z\rangle = \left(-z + x\sqrt{2} - \frac{p-1}{2z}\right)\langle x|z\rangle + \left(\frac{p-1}{2z}\right)\langle x|-z\rangle. \tag{17}$$

This has the solution

$$\langle x|z\rangle = N_0 \exp\left(-\frac{z^2}{2} - \frac{x^2}{2}\right)E(\sqrt{2}xz) \tag{18}$$

with N_0 a normalization constant. Substituting this into (14) gives

$$|x\rangle = N_0 \mathrm{e}^{-x^2/2}\sum_{n=0}^{\infty}|n\rangle\frac{\sqrt{[n]!}}{2\pi\mathrm{i}}\oint_{C^*}\frac{\mathrm{d}z^*}{(z^*)^{n+1}}\mathrm{e}^{-(z^*)^2/2}E(\sqrt{2}xz^*). \tag{19}$$

But from [11], in the parabose coordinate representation

$$|x\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|x\rangle = N_0\mathrm{e}^{-x^2/2}\sum_{n=0}^{\infty}|n\rangle\frac{H_n^{(p)}(x)}{\sqrt{2^n[n]!}} \tag{20}$$

$$H_n^{(p)}(x) = [n]!\sum_{k=0}^{[n/2]'}\frac{(-)^k(2x)^{n-2k}}{k![n-2k]!} \tag{21}$$

where $[k]'$ denotes the largest integer less than or equal to k. So since the $|n\rangle$ are complete,

$$H_n^{(p)}(x) = \frac{[n]!}{2\pi\mathrm{i}}\oint_C\frac{\mathrm{d}z}{(z)^{n+1}}\mathrm{e}^{-(z)^2/2}E(\sqrt{2}xz). \tag{22}$$

3. Eigenstates of the parafermi $f^\dagger$ operator

In the finite-dimensional Hilbert space of a single-mode parafermi system, the number states can be written as

$$|n\rangle = \frac{(f^\dagger)^n}{\sqrt{\{n\}!}}|0\rangle \qquad N_f|n\rangle = n|n\rangle \tag{23}$$

where $N_f = \frac{1}{2}[f^\dagger, f] + \frac{p}{2}$ is the parafermi number operator. Here

$$\{n\} = n(p+1-n) \qquad \{n\}! = \{n\}\{n-1\}\cdots\{1\} \qquad \{0\}! \equiv 1 \tag{24}$$

with n an integer, $0 \leqslant n \leqslant p$. In this basis, $f^\dagger|n\rangle = \sqrt{\{n+1\}}|n\rangle$, $f|n\rangle = \sqrt{\{n\}}|n-1\rangle$. Since $f^\dagger|p\rangle = 0$, there is the useful fact that

$$|n\rangle = \sqrt{\frac{\{n\}!}{\{p\}!}}\, f^{p-n}\,|p\rangle. \tag{25}$$

To describe the parafermi eigenstates of f (and of $f^\dagger$) in this number basis, we use [6] paragrassman numbers ξ obeying $\xi^{p+1} = 0$. The unnormalized eigenstate of the parafermi annihilation operator f

$$|\xi\rangle = \sum_{n=0}^{p}|n\rangle\frac{\xi^n}{\sqrt{\{n\}!}} \tag{26}$$

satisfies the eigenequation $f|\xi\rangle = |\xi\rangle\xi$. In this formulation, $|\xi\rangle$ is expandable in the number basis, cf [6]. Note that ξ stands to the right of $|\xi\rangle$. The overlap of two eigenstates $|\xi\rangle$ and $|\zeta\rangle$ is

$$\langle\xi|\zeta\rangle = \sum_{n=0}^{p}\frac{(\xi^*)^n\zeta^n}{\{n\}!} \tag{27}$$

where ξ^* is the conjugate of ξ. By the paragrassmann integral formula (see appendix A)

$$\int \xi^n \, \mathrm{d}\mu(\xi, \xi^*) \; (\xi^*)^m = \delta_{n,m} \; \{n\}! \tag{28}$$

and (26), there is the resolution of unity

$$\begin{aligned} &\int |\xi\rangle \mathrm{d}\mu(\xi, \xi^*)\langle\xi| = I \\ &\mathrm{d}\mu(\xi, \xi^*) = \mathrm{d}^p\xi^* \, \mathrm{d}^p\xi \; \exp\!\left(-\tfrac{1}{2}[\xi^*, \xi]\right). \end{aligned} \tag{29}$$

To constuct the eigenstates of the parafermi creation operator $f^\dagger$, we recall (25) and consider

$$|\xi\rangle' = \sum_{n=0}^{p} |n\rangle(-\xi^*)^{p-n}\sqrt{\frac{\{n\}!}{\{p\}!}}. \tag{30}$$

These are the desired eigenstates, since

$$\begin{aligned} f^\dagger|\xi\rangle' &= \sum_{n=0}^{p-1} |n+1\rangle(-\xi^*)^{p-n}\sqrt{\frac{\{n+1\}!}{\{p\}!}} = \sum_{n=1}^{p} |n\rangle(-\xi^*)^{p-n+1}\sqrt{\frac{\{n\}!}{\{p\}!}} \\ &= -|\xi\rangle' \, \xi^*. \end{aligned} \tag{31}$$

The overlap of these eigenstates is

$$'\langle\xi|\zeta\rangle' = \sum_{n=0}^{p} \frac{\{n\}!}{\{p\}!}\xi^{p-n}(\zeta^*)^{p-n} = \sum_{n=0}^{p} \frac{\{p-n\}!}{\{p\}!}\xi^n(\zeta^*)^n. \tag{32}$$

As for the f eigenstates in (29), the eigenstates $|\xi\rangle'$ obey a resolution of unity

$$\int |\xi\rangle' \mathrm{d}\mu(\xi^*, \xi) \;\, '\langle\xi| = I \tag{33}$$

where

$$\mathrm{d}\mu(\xi^*, \xi) = \mathrm{d}^p\xi \, \mathrm{d}^p\xi^* \; \exp\!\left(-\tfrac{1}{2}[\xi, \xi^*]\right). \tag{34}$$

This follows from (26) and (30) by

$$\begin{aligned} &\int |\xi\rangle' \mathrm{d}\mu(\xi^*, \xi) \;\, '\langle\xi| \\ &\quad = \sum_{n,m=0}^{p} |n\rangle \int (-\xi^*)^{p-n} \, \mathrm{d}\mu(\xi^*, \xi)(-\xi)^{p-m}\langle m| \; \frac{\sqrt{\{n\}!\{m\}!}}{\{p\}!} \\ &\quad = \sum_{n=0}^{p} \frac{\{n\}!}{\{p\}!}\{p-n\}!|n\rangle\langle n| = I \end{aligned} \tag{35}$$

where we have used the fact that $\{n\}! = n! \; p!/(p-n)!$.

Furthermore, with the aid of the differentiation formula (see appendix B)

$$\frac{\partial}{\partial\xi} \, \xi^n = \{n\} \, \xi^{n-1} = \xi^n \, \frac{\overleftarrow{\partial}}{\partial\xi} \qquad (0 \leqslant n \leqslant p) \tag{36}$$

we have

$$f^\dagger \, |\xi\rangle = |\xi\rangle \, \frac{\overleftarrow{\partial}}{\partial\xi} \qquad f \, |\xi\rangle' = -|\xi\rangle' \, \frac{\overleftarrow{\partial}}{\partial\xi^*} \tag{37}$$

which give the matrix elements of $f^\dagger$, f:

$$\langle\xi|\ f^\dagger\ |\xi\rangle' = \xi^*\ \langle\xi|\xi\rangle' = -\ \langle\xi|\xi\rangle'\ \xi^*$$
$$\langle\xi|\ f\ |\xi\rangle' = \frac{\partial}{\partial\xi^*}\ \langle\xi|\xi\rangle' = -\langle\xi|\xi\rangle'\ \frac{\overleftarrow{\partial}}{\partial\xi^*}. \tag{38}$$

Alternatively, these equations follow from (26) and (30), since

$$\langle\xi|\xi\rangle' = \frac{(-)^p}{\sqrt{\{p\}!}}\sum_{n=0}^{p}(-)^n\ (\xi^*)^n(\xi^*)^{p-n}. \tag{39}$$

Lastly, as in the parabose case (13), there is a contour-like form resolution of unity for the $f^\dagger$ and f eigenstates:

$$\int|\xi\rangle'\ \mathrm{d}^p\xi^*\ \langle\xi| = \sum_{n,m=0}^{p}\sqrt{\frac{\{n\}!}{\{p\}!}}|n\rangle\int(-\xi^*)^{p-n}\ \mathrm{d}^p\xi^*\ \ (\xi^*)^m\langle m|\ \frac{1}{\sqrt{\{m\}!}}$$
$$= \sum_{n=0}^{p}|n\rangle\langle n| = I \tag{40}$$

where we have used

$$\int\mathrm{d}^p\xi^*\,(\xi^*)^p = p! \qquad \int\ \mathrm{d}^p\xi^*\,(\xi^*)^n = 0 \quad (0\leqslant n\leqslant p) \tag{41}$$

and $\xi^*\,\mathrm{d}^p\xi^* = -\mathrm{d}^p\xi^*\,\xi^*$. Note that in (40), as in the parabose case (13), the integration is only over a single variable in the contour form of the resolution of unity, whereas in (29) and (35) it is over two variables as for the usual parabose coherent states.

4. Conserved-charge parabose creation operator eigenstates for order $p = 2$

The parabose creation and annihilation operators for the two-mode system satisfy the trilinear commutation relations

$$[a_k, \{a_l^\dagger, a_m\}] = 2\delta_{kl}a_m \qquad [a_k, \{a_l^\dagger, a_m^\dagger\}] = 2\delta_{kl}a_m^\dagger + 2\delta_{km}a_l^\dagger$$
$$[a_k, \{a_l, a_m\}] = 0 \qquad (k, l, m = 1, 2) \tag{42}$$

where $a_1 = a$, $a_2 = b$. Since $ab \neq ba$ for $p \geqslant 2$, there is a degeneracy in the states with n parabosons a and m parabosons b. For such states, we find [9] that the degree of degeneracy is 'min$(n, m) + 1$'. The complete set of state vectors† is

$$|n, m; i\rangle = \frac{1}{\sqrt{N_{n,m}^i}}(a^\dagger)^{n-i+S}(b^\dagger)^{m-2[(i-S)/2]}(a^\dagger b^\dagger)^{2[(i-S)/2]}(a^\dagger)^{i-S-2[(i-S)/2]}|0\rangle \tag{43}$$

where $N_{n,m}^i$ is the normalization constant, and $S = \frac{1}{2}(1-(-)^m)$, and i is the degeneracy index $1 \leqslant i \leqslant \min(n, m) + 1$. For parastatistics of order $p = 2$, the $\{|n, m; i\rangle\}$ are an orthonormal set basis vectors with normalization constant

$$(N_{n,m}^i)^2 = 2^{n+m}\left[\frac{n+i}{2}\right]!\left[\frac{n+1-i}{2}\right]!\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!\,. \tag{44}$$

In this basis, $a^\dagger, b^\dagger, a, b$ also act as raising and lowering operators (the explict fomulae are given in [9, equations (15)–(18)]).

† Note that here in section 4 (but not in section 2) $[x]$ denotes the integer part of x for $x \geqslant 0$.

If we consider a and b to be two types of parabose quanta possessing Abelian charges '+1' and '−1', then the charge operator is

$$Q \equiv N_a - N_b \tag{45}$$

with $N_a = \frac{1}{2}\{a^\dagger, a\} - 1$, $N_b = \frac{1}{2}\{b^\dagger, b\} - 1$. This charge operator Q commutes with the operators $a^\dagger b^\dagger$ and $b^\dagger a^\dagger$, so their common eigenstate should satisfy the eigenequations

$$\begin{aligned} &Q\,|q,z,w\rangle' = q|q,z,w\rangle' \\ &a^\dagger b^\dagger|q,z,w\rangle' = w^*|q,z,w\rangle' \qquad b^\dagger a^\dagger|q,z,w\rangle' = z^*|q,z,w\rangle'. \end{aligned} \tag{46}$$

Expanding $|q,z,w\rangle'$ in terms of the complete set of orthonormal basis vectors $|n,m;i\rangle$ for the two-mode parabose system, for $q \geqslant 0$ we have from the Q eigenequation (46):

$$|q,z,w\rangle' = \sum_{m=0}^{\infty}\sum_{i=1}^{m+1} c^i_{q+m,m}\;|q+m,m;i\rangle. \tag{47}$$

From the remaining two eigenequations, we obtain the coefficients

$$\begin{aligned} c^i_{q+m,m} &= \frac{(-)^m 2^m \sqrt{\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}}{\sqrt{\left[\frac{q}{2}\right]!\left[\frac{q+1}{2}\right]!\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!}}\,\delta^{(s)}(w^*)\delta^{(r)}(z^*) \\ &= \frac{1}{\sqrt{\left[\frac{q}{2}\right]!\left[\frac{q+1}{2}\right]!}}\;\frac{2^m\sqrt{\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}}{(w^*)^{1+s}(z^*)^{1+r}}\Bigg|_{C^*,B^*} \end{aligned} \tag{48}$$

where the integers

$$r \equiv \left[\frac{m-(-)^{q+m+i}i}{2} + \frac{1-(-)^q}{4}\right]$$

$$s \equiv \left[\frac{m+(-)^{q+m+i}i}{2} + \frac{1+(-)^q}{4}\right].$$

The anticlockwise contours C^* and B^* enclose respectively the origins in the complex z^* and w^* planes. Since for a specific q-sector the overall $([\frac{q}{2}]![\frac{q+1}{2}]!)^{-1/2}$ is constant, we omit it in the following analysis.

We list results only for the $q \geqslant 0$ sector: in it the unnormalized dual vectors are

$$|q,z,w\rangle' = \sum_{m=0}^{\infty}\sum_{i=1}^{m+1}\frac{2^m\sqrt{\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}}{(w^*)^{1+s}(z^*)^{1+r}}|q+m,m;i\rangle\Bigg|_{C^*,B^*} \tag{49}$$

whereas the unnormalized parabose conserved-charge coherent states themselves [9, 12] are

$$|q,v,u\rangle = \sum_{m=0}^{\infty}\sum_{i=1}^{m+1}\frac{v^r u^s}{2^m\sqrt{\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}}|q+m,m;i\rangle. \tag{50}$$

The inner product of $|q,z,w\rangle'$ and $|q,v,u\rangle$ is

$$\begin{aligned} \langle q,v,u|q,z,w\rangle' &= \sum_{m=0}^{\infty}\sum_{i=1}^{m+1}\left(\frac{v^*}{z^*}\right)^r\left(\frac{u^*}{w^*}\right)^s\left(\frac{1}{z^*w^*}\right)\Bigg|_{C^*,B^*} \\ &= \frac{1}{z^*-v^*}\Bigg|_{C^*}\;\frac{1}{w^*-u^*}\Bigg|_{B^*} \\ &= \delta(z^*-v^*)\,\delta(w^*-u^*) \qquad (|z^*|>|v^*|, |w^*|>|u^*|). \end{aligned} \tag{51}$$

These satisfy the contour form of the resolution of unity

$$\oint_{C^*}\oint_{B^*}\frac{dz^*}{2\pi i}\frac{dw^*}{2\pi i}\;|q,z,w\rangle'\,\langle q,z,w| = \sum_{m=0}^{\infty}\sum_{i=1}^{m+1}|q+m,m;i\rangle\,\langle q+m,m;i| = I_q \tag{52}$$

where I_q is the unity operator in the $q \geqslant 0$ sector.

In summary, working in the number basis, in this paper we construct the creation operator eigenvectors for single-mode parabosons and parafermions, and for the two-mode conserved-charge parabosons. The contour forms of the associated resolutions of unity are obtained.

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Appendix A. Proof of the paragrassman integration formula

We write $\xi = \sum_{i=1}^{p} \xi_i$, where the Green components ξ_i satisfy the relations

$$\{\xi_i, \xi_i\} = 0 \qquad [\xi_i, \xi_j] = 0 \quad (i \neq j). \tag{A.1}$$

Also

$$\frac{1}{2}[\xi^*, \xi] = \sum_{i=1}^{p} \xi_i^*\xi_i \quad (\tfrac{1}{2}[\xi^*, \xi])^2 = 2!\sum_{i<j}\xi_i^*\xi_j^*\xi_i\xi_j$$
$$\cdots \quad (\tfrac{1}{2}[\xi^*, \xi])^n = n!\sum_{i_1<\cdots<i_n}\xi_{i_1}^*\cdots\xi_{i_n}^*\xi_{i_1}\cdots\xi_{i_n}$$
$$\cdots \quad (\tfrac{1}{2}[\xi^*, \xi])^p = p!\,\xi_1^*\cdots\xi_p^*\xi_1\cdots\xi_p \tag{A.2}$$

so

$$\exp\big(-\tfrac{1}{2}[\xi^*, \xi]\big) = 1 + \sum_i \xi_i\xi_i^* + \sum_{i<j}\xi_i\xi_j\xi_i^*\xi_j^*$$
$$+ \cdots + \sum_{i_1<\cdots<i_n}\xi_{i_1}\cdots\xi_{i_n}\xi_{i_1}^*\cdots\xi_{i_n}^* + \cdots + \xi_1\cdots\xi_p\xi_1^*\cdots\xi_p^*. \tag{A.3}$$

For paragrassman integration, we adopt

$$\int d^p\xi\;\xi_1\cdots\xi_p = 1 \qquad \int d^p\xi\;\xi_{i_1}\cdots\xi_{i_n} = 0 \quad (0 \leqslant n < p) \tag{A.4}$$

where $d^p\xi \equiv d\xi_1\cdots d\xi_p$. These give (41).

By (A.4), the integral $\int \xi^n\,d\mu(\xi,\xi^*)\,(\xi^*)^m$ with $d\mu(\xi,\xi^*) = d^p\xi\,d^p\xi^*\exp\big(-\frac{1}{2}[\xi^*,\xi]\big)$ is non-zero only when $n = m$. So in this integration, we identify

$$\xi^n(\xi^*)^n \sim (n!)^2\sum_{i_1<\cdots<i_n}\xi_{i_1}\cdots\xi_{i_n}\xi_{i_1}^*\cdots\xi_{i_n}^* \tag{A.5}$$

$$\xi^p(\xi^*)^p \sim (p!)^2\xi_1\cdots\xi_p\xi_1^*\cdots\xi_p^*. \tag{A.6}$$

In the sum in (A.5) there are a total of $\binom{p}{n}$ terms†, and each such term contributes only once in the paragrassman integration; thus

$$\int \xi^n\;d^p\xi\;d^p\xi^*\;\exp\big(-\tfrac{1}{2}[\xi^*,\xi]\big)\,(\xi^*)^n = (n!)^2\binom{p}{n} = \frac{n!\,p!}{(p-n)!} = \{n\}!\,. \tag{A.7}$$

† Here $\binom{p}{n}$ denotes the ordinary binomial coefficient.

Appendix B. Proof of the paragrassman differentiation formula

The left-differentiation with respect to ξ is defined by

$$\frac{\partial}{\partial\xi} = \sum_{i=1}^{p} \frac{\partial}{\partial\xi_i} \tag{B.1}$$

where

$$\begin{aligned} &\left\{\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\xi_i}\right\} = 0 \qquad \left[\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\xi_j}\right] = 0 \quad (i \neq j) \\ &\left\{\frac{\partial}{\partial\xi_i}, \xi_i\right\} = 1 \qquad \left[\frac{\partial}{\partial\xi_i}, \xi_j\right] = 0 \quad (i \neq j). \end{aligned} \tag{B.2}$$

In terms of Green components, ξ^n can be expressed as

$$\xi^n = n! \sum_{i_1<\cdots<i_n} \xi_{i_1} \cdots \xi_{i_n} \tag{B.3}$$

where there are $\binom{p}{n}$ terms in the sum. For each j, in

$$\frac{\partial}{\partial\xi}\, \xi^n = n! \sum_{j=1}^{p} \frac{\partial}{\partial\xi_j} \left(\sum_{i_1<\cdots<i_n} \xi_{i_1} \cdots \xi_{i_n} \right) \tag{B.4}$$

there are $\binom{p-1}{n-1}$ terms in the inner summation which involve ξ_j and which survive after $\partial/\partial\xi_j$. So there are a total of $n!\ p\ \binom{p-1}{n-1}$ terms on the 'right-hand side' of (B.4) and they are of the form $\xi_{i_1} \cdots \xi_{i_{n-1}}$ with $i_1 < \cdots < i_{n-1}$. These are precisely the terms appearing in the Green-component expansion of ξ^{n-1}. By considering the symmetry of the Green components, we obtain the proportionality factor

$$\frac{n!\, p \binom{p-1}{n-1}}{(n-1)! \binom{p}{n-1}} = n(p+1-n) = \{n\} \tag{B.5}$$

so

$$\frac{\partial}{\partial\xi}\, \xi^n = \{n\}\, \xi^{n-1} \qquad (0 \leqslant n \leqslant p). \tag{B.6}$$

Right-differentiation can be dealt with in a similar manner.

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